A symmetry approach to exactly solvable evolution equations [a)]

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A method is developed for establishing the exact solvability of nonlinear evolution equations in one space dimension which are linear with constant coefficient in the highest-order derivative. The method, based on the symmetry structure of the equations, is applied to second-order equations and then to third-order equations which do not contain a second-order derivative. In those cases the most general exactly solvable nonlinear equations turn out to be the Burgers equation and a new third-order evolution equation which contains the Korteweg-de Vries (KdV) equation and the modified KdV equation as particular cases.

INTRODUCTION

This paper is concerned with the problem of determining whether a given nonlinear evolution equation is exactly solvable, and also with the problem of finding all such equations of a given order. An equation is called exactly solvable if it admits a Lax formulation,[1] that is, if there exist differential or integral operators L and A such that the given equation can be written in the form $L_t = [A, L]$.

Our approach to the above problem is based on the consideration of the symmetry structure of the given equation. If an equation is exactly solvable it possesses infinitely many generalized (as opposed to Lie-point) symmetries. The existence of a generalized symmetry manifests itself by the existence of an admissible generalized or Lie-Bäcklund (LB) operator.[2] The existence of infinitely many symmetries is expressed by the existence of a recursion operator Δ [3] (see also Sec. 1) which generates a new admissible LB operator from a given one. In almost[4] all known cases the admissible operator expressing invariance of equation under t-translation is generated from that expressing invariance under x-translation. We call equations possessing this property exactly solvable equations of normal type. Another interesting fact regarding the symmetry structure of evolution equations is that in all known cases the existence of one generalized symmetry implies the existence of infinitely many. (However, this has not been proved in general.) With the above in mind we now formulate our criterion, which is elaborated in Sec. 2.

A. Proposition

A necessary condition for a nonlinear evolution equation of nth order to be exactly solvable of normal type is that it admit an LB operator with a generator of order $2n-1$.

Having obtained this generator, it is usually possible by inspection to obtain the recursion operator Δ, the existence of which provides a sufficient condition for the exact solvability of this equation, since Δ and the Fréchet derivative of the t-independent part of the equation form a Lax pair (see Sec. 1).

The above criterion is quite practical since it is algorithmically very straightforward to find out if a given equation admits a generator of a given order. Further, it is also algorithmically possible to determine which equations of a certain order admit a generator of a given order. This is illustrated in Sec. 2, where we find all second-order equations and all third-order equations (not involving a second-order derivative) which are of normal type. Within equivalence (see Sec. 2.1), the most general nonlinear second-order equation with the above property is the Burgers equation. The most general third-order equations turn out to be: (i) A generalization of the Korteweg–de Vries (KdV) equation, see (2.18) which, in particular, contains any linear combination of the KdV and of the modified KdV (MKdV) as a special case, (ii) A linear combination of the potential KdV (PKdV) and of the potential MKdV (PMKdV). The potential KdV (or the potential version of the KdV) is the equation obtained from the KdV after replacing the dependent variable u by the "potential" w, $u = w_x$, and integrating once.

B. Outline of the paper

In Sec. 1 we first define admissible LB operators[2] as restricted to evolution equations as well as their commutators[5] and prove a lemma expressing the admissibility of an LB operator in a convenient form. We further recall the definition of a recursion operator[3] and then prove that the recursion operator together with the Fréchet derivative of the time-independent part of a given evolution equation form a Lax pair, and also give a convenient characterization of a recursion operator as well as of its main property (see Lemmas 2 and 3). Finally, for completeness of the presentation, the definition of a hereditary operator[6,7] is recalled. In Sec. 2 we first outline our method and then present some concrete results which also illustrate the general theory. In Sec. 2A we find all second-order equations possessing a third-order symmetry and the corresponding admissible generators. We further show that all these equations can be linearized, and give explicitly the corresponding linearizing Bäcklund transformations (BT). The recursion operators possessed by the above equations are also explicitly given. In Sec. 2.2 we find all third-order equations (not involving a second-order derivative) possessing a fifth-order symmetry. We also give the corresponding admissible generators and recursion operators. Finally, in Sec. 3 we compare our method with other existing ones and indicate some open questions.

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1. MATHEMATICAL PRELIMINARIES

A. Admissible LB operators

In what follows we shall consider evolution equations of the form

$$\Omega \equiv u_t + K(x,u,u_1,\ldots,u_n) = 0, \qquad (1.1)$$

where

$$u_j \equiv \left(\frac{\partial}{\partial_x}\right)^j u, \quad j = 0,1,\ldots,n. \qquad (1.2)$$

The most general LB operator associated with (1.1) is given by

$$X(\eta) \equiv \eta \frac{\partial}{\partial u} + (D_t \eta)\frac{\partial}{\partial u_t} + \sum_{j=1}^{\infty} (D^j \eta) \frac{\partial}{\partial u_j}, \qquad (1.3)$$

where $\eta = \eta(x,t,u,u_1,\ldots,u_N)$, N arbitrary, is called the generator of the above LB operator, D is the total derivative with respect to x

$$D \equiv D_x \equiv \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_{t1} \frac{\partial}{\partial u_t} + u_2 \frac{\partial}{\partial u_1} + \cdots,$$

and D_t is defined analogously. Without loss of generality, we assume that η does not depend on t-derivatives since for admissible operators they can always be eliminated using equation (1.1).

The LB operator $X(\eta)$ is an *admissible* LB operator of (1.1) iff $X(\eta)\Omega = \sigma$, where $\sigma = 0$ when Eq. (1.1) and its differential consequences are assumed. The above is denoted by

$$X(\eta)\Omega\,|_{\Omega = 0} = 0 \qquad (1.4)$$

Equation (1.4) provides an algorithm for finding η as the solution of a system of liner overdetermined equations.

An important special class of LB operators is the class of Lie (point) operators. The most general such operator associated with (1.1) is given by

$$Z = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial u}, \qquad (1.5)$$

where ξ, τ, ν are functions of x, t, and u only. The operator Z can be written in the form (1.3) by the equivalence[8]

$$Z \Leftrightarrow X(\nu - \xi u_1 - \tau u_t), \qquad (1.6)$$

or, using Eq. (1.1), $X(\nu - \xi u_1 + \tau K)$. The above equivalence means that Eq. (1.1) admits Z iff it admits the corresponding X.

The commutator of two LB operators is an LB operator whose generator is expressed by a simple formula,[5]

$$[X(\eta_1), X(\eta_2)] = X(\eta_3),$$

$$\text{where } \eta_3 = X(\eta_1)\eta_2 - X(\eta_2)\eta_1. \qquad (1.7)$$

Obviously, the admissible LB operators of a given equation form a Lie algebra.

In considering the symmetries of an equation, it is convenient to use an operator formulation.[9] We define the Fréchet derivative of a function $N(u) \equiv N(u,u_t,u_1,\ldots,u_n)$ by $N'(u)$, where

$$N'(u)[v] \equiv \frac{\partial N(u + \epsilon v)}{\partial \epsilon}\bigg|_{\epsilon = 0}. \qquad (1.8)$$

Clearly, the right-hand side of Eq. (1.8) is linear in v and therefore $N'(u)$ is an operator acting linearly on v. Actually,

$$N'(u) = \frac{\partial N}{\partial u} + \frac{\partial N}{\partial u_t} D_t + \sum_{j=1}^{n} \frac{\partial N}{\partial u_j} D^j. \qquad (1.9)$$

Comparing Eq. (1.3) and (1.9) we obtain

$$X(\eta)\Omega = \Omega'[\eta]. \qquad (1.10)$$

Therefore Eq. (1.4) takes the form

$$\Omega'[\eta]\,|_{\Omega = 0} = 0. \qquad (1.11)$$

In the case of evolution equations, Eq. (1.11) [or, equivalently, (1.4)] can be further simplified

Lemma 1: The evolution equation (1.1) admits the LB operator $X(\eta)$ generated by $\eta = \eta(x,t,u,u_1,\ldots,u_n)$ iff

$$\eta_t + X(\eta)K - X(K)\eta = 0 \qquad (1.12a)$$

or, equivalently,

$$\eta_t + K'[\eta] - \eta'[K] = 0. \qquad (1.12b)$$

Proof: Writing out Eq. (1.4), we obtain

$$(D_t\eta) + X(\eta)K = 0, \text{ when (1.1) holds,}$$

or

$$\eta_t + \frac{\partial \eta}{\partial u} u_t + \sum_{j=1}^{N} \frac{\partial \eta}{\partial u_j} u_{jt} + X(\eta)K = 0, \text{ when (1.1) holds,}$$

or

$$\eta_t - \frac{\partial \eta}{\partial u} K - \sum_{j=1}^{N} \frac{\partial \eta}{\partial u_j} D^j K + X(\eta)K = 0. \qquad \text{Q.E.D.}$$

The advantage of (1.12) in either form as compared with (1.4) [or (1.11)] is that the validity of (1.1) has already been assumed. Therefore the admissibility of $X(\eta)$ by (1.1) is expressed as a relation between η and K with no further assumptions to be made. Further, using Eq. (1.7) we obtain

Corollary 1: If η does not depend on t explicitly, Eq. (1.1) admits $X(\eta)$ iff $X(\eta)$ and $X(K)$ commute.

Recursion Operators: The operator $\Delta(u)$ is a recursion operator for Eq. (1.1) iff[3]

$$[\Omega', \Delta]_{\Omega = 0} = 0. \qquad (1.13)$$

It follows from the above definition that if $X(\eta)$ is an admissible LB operator of (1.1) and Δ is a recursion operator of (1.1), then the infinitely many LB operators $X(\Delta^j\eta)$, $j = 0, 1, 2,\cdots$, are also admissible operators for the equation (1.1).[3] (See also Lemma 3 and its corollary.)

We define the Fréchet derivative of an operator-valued function $\Delta(u)$ by

$$\Delta'(u)[v]w \equiv \frac{\partial[\Delta(u + \epsilon v)w]}{\partial \epsilon}\bigg|_{\epsilon = 0}, \qquad (1.14)$$

and say that $\Delta'(u)[v]w$ is the derivative of $\Delta(u)$ evaluated at v and then applied to w. For example, the recursion operator of the KdV equation is given by $\Delta(u) = D^2 + 2/3u + 1/3u_1 D^{-1}$, where D^{-1} is the inverse total derivative $D^{-1}(w)(x) = \int_a^x w(\xi)d\xi$. Therefore $\Delta'(u)[v] = 2/3v + 1/3\, v_1 D^{-1}$.

Lemma 2: The operator $\Delta(u)$ is a recursion operator for

Eq. (1.1) iff the operators $\Delta(u)$ and $K'(u)$ form a Lax pair for Eq. (1.1). This is a consequence of the following equivalence

$$[\Omega',\Delta]_{\Omega=0}=(\Delta_t+[K',\Delta])_{\Omega=0}=-\Delta'[K]+[K',\Delta]. \tag{1.15}$$

(Here Δ_t actually means $D_t\Delta$ whereas, for instance, in (1.12) η_t means $\partial\eta/\partial t$. The reason for this regrettable inconsistency of notations is that in discussing a Lax pair $D_t L$ is customarily denoted by L_t.)

Proof:

$$[\Omega',\Delta]\eta=[D_t+K',\Delta]\eta$$
$$=[D_t,\Delta]\eta+[K',\Delta]\eta=\Delta_t\eta+[K',\Delta]\eta,$$

since

$$[D_t,\Delta]\eta=D_t(\Delta\eta)-\Delta(D_t\eta)=\Delta_t\eta.$$

The second equivalence in (1.15) follows from the above using the chain rule of differentiation and Eq. (1.1), since $\Delta_t=\Delta'[u_t]=-\Delta'[K]$. Q.E.D.

The most convenient characterization of a recursion operator follows from the equation

$$\Delta'[K]=[K',\Delta], \tag{1.16}$$

since Eq. (1.1) has now been eliminated.

The following lemma expresses a useful property of a recursion operator.

Lemma 3: The operator Δ is a recursion operator of Eq. (1.1) iff

$$K'[\Delta\zeta]-(\Delta\zeta)'[K]=\Delta(K'[\zeta]-\zeta'[K]), \tag{1.17}$$

where $\zeta(x,t,u,u_1,...,u_n)$ is an arbitrary function of the arguments indicated.

Proof: Using Leibnitz's rule, we obtain

$$(\Delta\zeta)'[K]=\Delta'[K]\zeta+\Delta(\zeta'[K]).$$

Then using Eq. (1.16) we obtain that

$$(\Delta\zeta)'[K]=K'[\Delta\zeta]-\Delta(K'[\zeta])+\Delta(\zeta'[K])$$

iff Δ is a recursion operator of Eq. (1.1). Q.E.D.

From the above lemma and Eq. (1.12b) one finds

Corollary 2: If $\Delta(u)$ is a recursion operator [of Eq. (1.1)] not depending explicitly on t and $X(\eta)$, $\eta(x,t,u,u_1,...,u_N)$ is admitted by Eq. (1.1), then the LB operators $X(\Delta^j\eta)$, $j=1,2,\cdots$, are also admitted by (1.1).

Hereditary operators: Assume that Eq. (1.1) possesses a recursion operator Δ. We call hierarchy 1 the hierarchy of admissible operators

$$X(\Delta^j u_1),\ \ j=0,1,2,\cdots, \tag{1.18}$$

which are generated from the x-translation operator $X(u_1)$. It is obvious that the operators $X(u_t+\Delta^j u_1)$ are also admissible. Equating to zero the generators of these admissible LB operators we obtain

$$u_t+\Delta^j u_1=0,\ \ j=1,2,\cdots, \tag{1.19}$$

which is the Lax hierarchy of equations associated with Eq. (1.1).[1] In Ref. 6 it is shown that the operator Δ is a recursion operator of the whole hierarchy (1.19) if Δ satisfies

$$[\Delta,\Delta'][v]w=[\Delta,\Delta'][w]v, \tag{1.20}$$

where

$$[\Delta,\Delta'][v]w=\Delta(\Delta'[v]w)-\Delta'[\Delta v]w, \tag{1.21}$$

and v, w are arbitrary functions of u, $u_1,...,u_N$. An operator Δ satisfying the above property is called a hereditary operator.[6] It is clear that any operator Δ is a recursion operator for the equation $u_t+u_1=0$, since $\Delta[u_1]=[D,\Delta]$. Therefore, any hereditary operator Δ is a recursion operator for the whole hierarchy $u_t+\Delta^j u_1=0$ generated by this operator. Therefore, an alternative way to find a recursion operator Δ of Eq. (1.1) is to look for a Δ such that (i) Δ is hereditary and (ii) $\Delta u_1=K$. The above property of Δ was first introduced in Ref. (7), where it was used to prove that such a Δ generates the exactly solvable equations $u_t+C(\Delta)u_1=0$, where $C(Z)$ is an arbitrary function of Z, regular, except possibly at $|Z|\to\infty$ and some points Z_c ($Z_c<\infty$).

2. A METHOD FOR FINDING OUT IF A GIVEN EQUATION IS EXACTLY SOLVABLE

If an evolution equation admits a Lax formulation it also admits infinitely many symmetries. Actually, every member of the Lax hierarchy [see Eq. (1.19)] associated with a given solvable equation is a generator of a generalized symmetry admitted by this equation. Therefore, in order to establish that an equation is exactly solvable we must prove that it possesses infinitely many symmetries. Although there exists an algorithmic way of finding out if a function of the general form $\eta=\eta(u,u_1,...,u_N)$ is an admissible generator, this does not lead to a very practical method for establishing the existence of infinitely many symmetries. However, in all known cases the existence of one generalized symmetry seems to be sufficient for the existence of infinitely many. Further, having obtained one generalized symmetry it is usually possible, almost by inspection, to find a recursion operator Δ which generates infinitely many symmetries. Therefore, the problem of finding out if an equation is exactly solvable reduces to finding an LB symmetry.

In order to find an LB symmetry we must assume the order of the highest derivative in $\eta(u,u_1,...,u_N)$. But how can we know N *a priori*? It is at this point that we use the *existence* of Δ. The only assumption we make is that this Δ generates the t-translation symmetry of the equation from the x-translation symmetry. Let us be more specific. Suppose we are given an evolution equation of the form

$$u_t+u_n+\widetilde{K}(u,u_1,...,u_{n-1})=0. \tag{2.1}$$

This equation possesses two Lie-point generators, $\eta_1=u_1$ and $\eta_2=u_n+\widetilde{K}(u)$. If there exists a Δ which generates η_2 from η_1, then $\Delta=D^{n-1}+\cdots$. Therefore, the first LB generator is of the form $\eta_3=u_{2n-1}+g(u,u_1,..,u_{2n-2})$. That is, $N=2n-1$ and, furthermore, u_N appears linearly. The above discussion justifies, in our opinion, the proposition made in the introduction.

A. Finding all second-order equations which possess a third order symmetry

In this subsection we first determine all equations of the form

$$u_t+u_2+A(u,u_1)=0 \tag{2.2}$$

possessing an admissible generator of the form

$$\eta = u_3 + B(u,u_1,u_2). \tag{2.3}$$

It turns out that all equations having this property also possess infinitely many symmetries and further, all can be linearized.

The following equations and corresponding generators are obtained (for details see Appendix A)

(i)

$$u_t + u_2 + \frac{b''(u)}{b'(u)} u_1^2 + \alpha b(u) u_1 = 0, \tag{2.4a}$$

$$\eta = u_3 + \frac{b''}{b'} u_1^3 + \frac{3b''}{b'} u_1 u_2 + \tfrac{3}{2}\alpha\left(b' + \frac{bb''}{b'}\right) u_1^2 + \tfrac{3}{2}\alpha b u_2 + \tfrac{3}{4}\alpha^2 b^2 u_1, \tag{2.4b}$$

where $b(u)$ is an arbitrary function of u, $b'(u) = db/du$, and α is an arbitrary parameter. (Everywhere in this paper greek lower-case letters stand for constant parameters.)

(ii)

$$u_t + u_2 + \left[\frac{\gamma - c'(u)}{c(u)}\right] u_1^2 + \alpha c(u) = 0, \tag{2.5a}$$

$$\eta = u_3 + \left[\left(\frac{\gamma - c'}{c}\right)^2 + \left(\frac{\gamma - c'}{c}\right)'\right] u_1^3 + 3\left(\frac{\gamma - c'}{c}\right) u_1 u_2, \tag{2.5b}$$

where $c(u)$ is an arbitrary function of u. For the discussion to follow it is convenient to let $c \equiv d/d'$, $d \equiv d(u)$. Then (2.5a) becomes

$$u_t + u_2 + [d''/d' + (\gamma - 1)d'/d]u_1^2 + \alpha d/d' = 0. \tag{2.5c}$$

We can add a constant multiple of u_1 to the left-hand side of (2.4a) and (2.5a) without altering the above results. This has been omitted for economy of writing.

We define two equations to be equivalent if one can be obtained from the other by a transformation involving only the dependent variable. Then it is clear that Eq. (2.4a) is equivalent to

$$u_t + u_2 + \alpha u u_1 = 0 \tag{2.6}$$

under the transformation $u \to b(u)$, while Eq. (2.5c) is equivalent to

$$u_t + u_2 + \alpha\gamma u = 0 \tag{2.7}$$

under the transformation $u \to [d(u)]^{\gamma}$. Therefore, within equivalence the only nonlinear equation of the form (2.2) admitting a generalized symmetry of the form (2.3) is the Burgers equation. Note that under an equivalence transformation Eqs. (2.4a) and (2.5a) remain exactly solvable and must hence retain the same form.

1. Linearizing Bäcklund Transformations

It turns out that all the above equations can be linearized. Also, if they are the only second-order equations exactly solvable, then they are the only equations of the general form (2.2) which can be linearized. The following results are obtained in Ref. 10, Sec. 5.4.1:

(i) The only equation of the form (2.2) mapped to $v_t + v_2 = 0$ under a BT of the general form $v_1 - f(u,v) = 0$ is given by (2.4a). This BT takes the form

$$b(u) = 2v_1/(\alpha v + \lambda). \tag{2.8}$$

(ii) The only equation of the form (2.2) mapped to $v_t + v_2 = 0$ under a BT of the form $u_1 - f(u,v) = 0$ is given by (2.5a) with $\gamma \equiv 0$. This BT takes the form

$$v = (u_1/\alpha c(u)). \tag{2.9}$$

(iii) The only equation of the form (2.2) mapped to $v_t + v_2 + \alpha\gamma v = 0$ under a map of the form $u = f(v)$ is given by (2.5c). This map is

$$d(u) = v^{1/\gamma}. \tag{2.10}$$

Every linear equation possesses infinitely many symmetries. Therefore, every nonlinear equation which can be linearized also possesses infinitely many symmetries. However, the reverse is not true; that is, not every equation possessing infinitely many symmetries can be linearized (for example, the KdV). In the case of second-order equations, however, we see that the class of equations possessing a generalized symmetry (and actually infinitely many, see below) coincides with the class of second-order equations which can be linearized.

2. Recursion Operators

Equations (2.4a) and (2.5a) possess, respectively, the following recursion operators

$$\Delta = D + \frac{b''}{b'} u_1 + \tfrac{1}{2}\alpha b + \tfrac{1}{2}\alpha u_1 D^{-1}(b' .), \tag{2.11}$$

$$\Delta = D + \left(\frac{\gamma - c'}{c}\right) u_1. \tag{2.12}$$

The operator (2.11) reduces to

$$\Delta = D + \tfrac{1}{2}\alpha u + \tfrac{1}{2}\alpha u_1 D^{-1} \tag{2.13}$$

when $b = u$, which is known to be the recursion operator of the Burgers equation.[3]

It is easy to check that the operator Δ defined by (2.11) is a hereditary operator. Further, $\Delta^2(u_1)$ is the generator (2.4b). Consider, now, the operator (2.12). It can be shown easily that $\Delta = D + a(u)u_1$, where $a(u)$ is an arbitrary function of u, is a hereditary operator. Therefore, Δ is a recursion operator for the equation $u_t + \Delta u_1 = 0$ or $u_t + u_2 + a u_1^2 = 0$. Further, it is clear that the above operator will also be a recursion operator of the equation $u_t + u_2 + a u_1^2 + c = 0$, where $c(u)$ is an arbitrary function of u, iff it is a recursion operator of the equation $u_t + c = 0$, that is iff [see (1.16)]

$$\Delta'[c] = [c', \Delta].$$

This implies $c'' + (ac)' = 0$ or $a = (\gamma - c')/c$. Therefore, Δ is a recursion operator of

$$u_t + u_2 + a u_1^2 + c(u) = 0, \tag{2.14a}$$

iff

$$a = (\gamma - c')/c. \tag{2.14b}$$

Note that since Δ is a recursion operator of (2.14a), where a is given by (2.14b), $\Delta(u_1) = u_2 + a u_1^2$ is a symmetry generator of (2.14a) and, since the whole right-hand side of (2.14a) is also a symmetry, it follows that $c(u)$ is also. This can be trivially checked directly. Also $\Delta c = \gamma u_1$ and therefore the generator $c(u)$ does not produce a new hierarchy of symmetries.

In Ref. 11 (which is an excellent exposition of the Estabrook–Wahlquist method) Kaup asks which equations of the general form $u_t + u_2 + u_1^2 + f(u) = 0$ possess a nontrivial prologation structure. He then finds that $f(u) = \beta e^{-u} + \gamma$ and also develops a method of solving the above equation. Note that if $(\gamma - c')/c = 1$ in Eq. (2.5a), $c(u) = \beta e^{-u} + \gamma$ and, further, this equation is equivalent to a linear one; therefore it is trivially solved.

B. Finding all third-order equations, not involving second-order derivatives, which possess a fifth-order symmetry

In this section we determine all equations of the form

$$u_t + u_3 + A(u,u_1) = 0 \tag{2.15}$$

possessing an admissible generator of the form

$$\eta = u_5 + B(u,u_1,u_2,u_3,u_4). \tag{2.16}$$

The following equations and corresponding generators are obtained (for details see Appendix B).

(i)

$$u_t + u_3 + \alpha u_1^3 + \beta u_1^2 + \gamma u_1 = 0, \tag{2.17a}$$

$$\eta = u_5 + 5\alpha u_1^2 u_3 + \tfrac{10}{3}\beta u_1 u_3 + 5\alpha u_1 u_2^2 + \tfrac{5}{3}\beta u_2^2 + \tfrac{3}{2}\alpha^2 u_1^5 + \tfrac{10}{9}\beta^2 u_1^3 + \tfrac{5}{2}\alpha\beta u_1^4. \tag{2.17b}$$

(ii)

$$u_t + u_3 + \alpha u_1^3 + b(u)u_1 = 0, \tag{2.18a}$$

where $b(u)$ solves

$$b''' + 8\alpha b' = 0, \tag{2.18b}$$

$$\eta = u_5 + 5\alpha u_1^2 u_3 + \tfrac{5}{3}bu_3 + 5\alpha u_1 u_2^2 + \tfrac{10}{3}b'u_1u_2 + \tfrac{3}{2}\alpha^2 u_1^5 + \tfrac{5}{3}\alpha b u_1^3 + \tfrac{5}{6}b''u_1^3 + \tfrac{5}{6}b^2 u_1. \tag{2.18c}$$

It is clear that Eq. (2.17a) is the potential version of the special case of (2.18a) where $\alpha = 0$. In this sense, the most general equation of the form (2.15) admitting a symmetry generator of the form (2.16) is given by (2.18).

Particular Cases:

(i)
$\alpha = 0$ in (2.17a) (PKdV),

$$\eta = u_5 + \tfrac{10}{3}\beta u_1 u_3 + \tfrac{5}{3}\beta u_2^2 + \tfrac{10}{9}\beta^2 u_1^3. \tag{2.19}$$

(ii)
$\beta = 0$ in (2.17a) (PMKdV),

$$\eta = u_5 + 5\alpha u_1^2 u_3 + 5\alpha u_1 u_2^2 + \tfrac{3}{2}\alpha^2 u_1^5. \tag{2.20}$$

(iii)
$b = 0$ in (2.18a) (PMKdV),

$$\eta = u_5 + 5\alpha u_1^2 u_3 + 5\alpha u_1 u_2^2 + \tfrac{3}{2}\alpha^2 u_1^5. \tag{2.21}$$

(iv)
$\alpha = 0, b = u$ in (2.18a) (KdV),

$$\eta = u_5 + \tfrac{5}{3}uu_3 + \tfrac{10}{3}u_1u_2 + \tfrac{5}{6}u^2u_1. \tag{2.22}$$

(v)
$\alpha = 0, b = u^2$ in (2.18a) (MKdV),

$$\eta = u_5 + \tfrac{5}{3}u^2u_3 + \tfrac{20}{3}uu_1u_2 + \tfrac{5}{3}u_1^3 + \tfrac{5}{6}u^4u_1. \tag{2.23}$$

1. Recursion Operators

Equations (2.17a) and (2.18a) possess, respectively, the following recursion operators

$$\Delta = D^2 + \gamma + 2\alpha u_1^2 + \tfrac{4}{3}\beta u_1 - 2\alpha u_1 D^{-1}(u_2\cdot) - \tfrac{2}{3}\beta D^{-1}(u_2\cdot), \tag{2.24}$$

$$\Delta = D^2 + 2\alpha u_1^2 + \tfrac{2}{3}b - 2\alpha u_1 D^{-1}(u_2\cdot) + \tfrac{1}{3}u_1 D^{-1}(b'\cdot). \tag{2.25}$$

Letting α or β equal zero in (2.24), we obtain the recursion operator of the PKdV or of the PMKdV, respectively. Letting $b = 0$ in (2.25), we obtain the recursion operator of the PMKdV. Letting $\alpha = 0$ in (2.25), we obtain the recursion operator of the linear combination of the KdV[3] and of the MKdV.[3] It is easily checked that both (2.24) and (2.25) are hereditary operators. Further, it is interesting that if we start with (2.25) and require that it is a hereditary operator we find out that this is the case iff b satisfies (2.18b). Equation (2.18b) also appears when applying Δ to $u_3 + \alpha u_1^3 + b(u)u_1$ in order to obtain the generator (2.18c). Let us consider only the terms involving integration

$$\tfrac{1}{3}u_1 D^{-1}(b'u_3 + \alpha b'u_1^3 + bb'u_1) - 2\alpha u_1 D^{-1}(u_2u_3 + \alpha u_2 u_1^2 + bu_2u_1).$$

The terms involving $bb'u_1$, $\alpha u_2 u_1^2$, and u_2u_3 integrate exactly and so we are left with

$$\tfrac{1}{3}u_1 D^{-1}(b'u_3 + \alpha b'u_1^3) - 2\alpha u_1 D^{-1}(bu_2u_1).$$

Integrating the first term by parts and ignoring the part integrated exactly, we are left with

$$-\tfrac{1}{3}u_1 D^{-1}[u_2u_1(b'' + 8\alpha b)],$$

which is exactly integrable iff Eq. (2.18b) holds.

2. A Bäcklund Transformation

It is well known tht the KdV equation is related to the MKdV equation through the Miura transformation. It is interesting that Eq. (2.18) is also related to the MKdV equation [trivially generalized, see (2.28) below]. Taking into consideration (2.18b), Eq. (2.18a) becomes

$$u_t + u_3 + \alpha u_1^3 + (\tau_1 e^{2\sqrt{-2\alpha u}} + \tau_2 e^{-2\sqrt{2\alpha u}} + \tau_3)u_1 = 0, \tag{2.26}$$

where τ_1, τ_2, τ_3 are constant parameters. The Bäcklund transformation

$$u_1 + \kappa v + \lambda + \left(\frac{\tau_1}{3\alpha}\right)^{1/2} e^{u\sqrt{2\alpha}} + \left(\frac{\tau_2}{3\alpha}\right)^{1/2} e^{-u\sqrt{2\alpha}} = 0, \tag{2.27}$$

where κ, λ are constant parameters, maps Eq. (2.26) to

$$v_t + v_3 + \left[3\alpha(\kappa v + \lambda)^2 + \tau_3 - \frac{1}{3\alpha}(\tau_1\tau_2)^{1/2}\right]v_1 = 0. \tag{2.28}$$

Note that if Eq. (2.27) is viewed as an ordinary differential equation with x and u as the independent and dependent variables, respectively, (t is regarded as a parameter) then it is of the Riccati type. If we put $u = (1/\sqrt{-2\alpha})\ln w$, (2.27) becomes

$$w_x + (-\tfrac{2}{3}\tau_2)^{1/2} + (-2\alpha)^{1/2}(\kappa v + \lambda)w + (-\tfrac{2}{3}\tau_1)^{1/2}w^2 = 0.$$

3. COMPARISON WITH OTHER METHODS AND OPEN QUESTIONS

The most obvious approach to establishing the exact solvability of a given equation is to guess operators L and A such that the given equation can be expressed in the form

$$L_t = [A,L]. \tag{3.1}$$

However, this approach is the least practical, since both operators A and L must be guessed. A way out is to assume the form of L and then find all equations that correspond to it. In this respect there exist two basic approaches; (i) Gel'fand and Dikii[12] assume L and then, by solving Eq. (3.1) algebraically, find all equations that correspond to it. (ii) AKNS[13] (see also Ref. 14) assume a given L, but rather than using Eq. (3.1) directly, they determine all equations corresponding to this L by requiring that the evolution of the scattering data takes a simple form. This method has been extended by Newell.[15] The above approach has the advantage that it also paves the way for the actual solution of the evolution equation involved, but has the weakness that it starts with a given eigenvalue problem and finds all equations that correspond to it, rather than starting directly with a given equation.

The most widely used direct method for finding whether a given equation is exactly solvable has been developed by Estabrook and Wahlquist.[16] This method consists, briefly, of the following (for consistence of presentation we do not use the language of differential forms employed in Ref. 16): Find functions $A(u, Q)$ and $B(u, u_1,\dots,u_{n-1}, Q)$ (the assumption made about the dependence of A and B is based on experience) such that the equations $Q_x = A$, $Q_t = B$ are compatible when u satisfies the given nth-order equation. This easily leads to $A = \Sigma_j a_j(u)\xi_j(Q)$, $B = \Sigma_j b_j(u,u_1,\dots,u_{n-1})\xi_j(Q)$, where the functions a_j and b_j are completely determined and the ξ_j satisfy some given commutator relations. The main problem now is to find a closed algebra of ξ_j and then a representation of this algebra in terms of Q. Also, sometimes it is necessary to allow Q to be a vector.

Another direct approach is introduced in Ref. 17, where A in Eq. (3.1) is taken to be the adjoint of $K'(u)$, and L is a recursion operator connecting polynomial solutions of the equation $\psi_t + A\psi = 0$. These solutions are simply related to the conservation laws of the given equation by a theorem due to Lax.[18] The authors of Ref. 17 employ a perturbative scheme to find conservation laws and then L. A weakness of this method is that it is applicable only to exactly solvable equations with infinitely many conservation laws. However, there exist exactly solvable equations with a finite number of conservation laws (for example, the Burgers equation).

The formalism of taking as a Lax pair $K'(u)$ and the operator Δ (which recursively relates solutions of the equation $\psi_t + K'(u)\psi = 0$) is developed in Ref. 7. However, this formalism was not directly related to the symmetry structure of the given equation. This is done (apparently independently) in Ref. 6; see also Sec. 1. The advantage of this approach is that Eq. (3.1) has to be solved only for L, since $A = K'(u)$ is explicitly known.

In this paper we emphasize that the knowledge of one generalized symmetry makes it possible to obtain Δ almost by inspection. That is why we concentrate on finding such a generalized symmetry; the relevant algorithm employed is very straightforward. Furthermore, demanding that an equation of a certain order admit a generator of a given order, we obtain in a straightforward way (the algorithm involved is linear) all such equations. Our method has the weakness that it is applicable to equations of normal type only. Furthermore, having obtained Δ, we must solve the equations

$$\Delta\psi = \lambda\psi, \quad \psi_t + K'(u)\psi = 0. \tag{3.2}$$

However, these equations are not in a very convenient form, since the first equation involves an integral operator. A proper transformation makes it possible to transform the above equations to differential ones (for example, in the case of the KdV this is achieved by taking $\psi = \phi^2$). A general method for doing this as well as an investigation of Eq. (2.18) will be presented in a future publication.

Apparently, there exists an intimate connection between our method and Estabrook-Wahlquist's one. By asking two different questions (namely, when a given equation has a nontrivial prologation structure and when it admits a generalized symmetry) we obtain similar answers[19] (see also Sec. 2.1). The problem of relating these two methods is under investigation. The problem of extending the results obtained here to equations of less restricted form is also under investigation. For example, results have been obtained for nonlinear heat equations.

We hope that the results presented here together with those of Refs. 20 and 21 (where the group-theoretical nature of BT and of the constants of motion admitted by evolution equations is established) as well as those of Refs. 22–26, indicate the importance played by symmetries in understanding and solving the problems appearing in the analysis of nonlinear evolution equations.

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APPENDIX A

In this appendix we indicate briefly how Eq. (2.4) and (2.5) are obtained. Equation (2.2) admits the LB operator associated with the generator (2.3) iff (see Lemma 1)

$$[X(u_2 + A(u,u_1)), X(u_3 + B(u,u_1,u_2))] = 0, \tag{A1}$$

or

$$\Big(D^3 + \sum_{j=0}^{2} B_{j+1}D^j\Big)(u_2 + A) = (D^2 + A_1 + A_2D)(u_3 + B), \tag{A2}$$

where

$$B_{j+1} = \partial B/\partial u_j,\ A_{j+1} = \partial A/\partial u_j, \quad j = 0,1,2,$$

or

$$D^2B + \sum_{j=0}^{1} A_{j+1} + D^jB + \sum_{1}^{2} A_j u_{j+2}$$
$$= D^3A + \sum_{j=0}^{2} B_{j+1} D^jA + \sum_{1}^{3} B_j u_{j+1}. \quad \text{(A3)}$$

Writing out $D^jA, D^jB, (j = 1,2,3)$ explicitly in Eq. (A3) and then equating to zero the coefficients of u_3^2 and u_3, we obtain

$$B_{33} = 0, \quad 2DB_3 = 3DA_2, \quad \text{(A4)}$$

or

$$B = \tfrac{3}{2}A_2u_2 + \alpha u_2 + F(u,u_1). \quad \text{(A5)}$$

The parameter α generates the t-translation group so we set it equal to zero. Now substituting (A5) in (A3) and equating to zero the coefficients of $u_2{}^j$ $(j = 3, 2, 1, 0)$, we obtain

$$A_{222} = 0, \quad F_{22} = \tfrac{3}{2}A_2A_{22} + 3A_{12},$$
$$2F_{12}u_1 = \tfrac{3}{2}AA_{12} + \tfrac{3}{2}A_1A_{22}u_1 + \tfrac{3}{2}A_2A_{12}u_1$$
$$+ \tfrac{3}{2}A_{112}u_1^2 + 3A_{11}u_1, \quad \text{(A6)}$$
$$A_1F + A_2F_1u_1 + F_{11}u_1^2 + AF_1 + A_1F_2u_1$$
$$+ \tfrac{3}{2}A_2A_{11}u_1^2 + A_{111}u_1^3 = 0.$$

Solving Eqs. (A6) and taking into consideration (A5), we obtain

(i)

$$A = a(u)u_1^2, \quad \text{(A7a)}$$

where a is an arbitrary function of u.

(ii)

$$A = \frac{b''}{b'}u_1^2 + bu_1, \quad \text{(A7b)}$$

where $b(u)$ is an arbitrary function of u and $b' = db/du$. Equations (A7) can be combined into one by letting $b \to \beta b$. Then

$$A = \frac{b''(u)}{b'(u)}u_1^2 + \beta b(u)u_1 \quad \text{(A8a)}$$

[and $\beta = 0$ gives (A7a)]. To the above A there corresponds

$$B = \frac{b'''}{b'}u_1^3 + \frac{3b''}{b'}u_1u_2 + \tfrac{3}{2}\beta\Big(b' + \frac{bb''}{b'}\Big)u_1^3$$
$$+ \tfrac{3}{2}\beta bu_2 + \tfrac{3}{4}\beta^2b^2u_1. \quad \text{(A8b)}$$

(iii)

$$A = (\gamma - c')/cu_1^2 + c, \quad \text{(A9)}$$

where c is an arbitrary function of u, B is given by (2.5b).

APPENDIX B

In this appendix we indicate briefly how Eqs. (2.17) and (2.18) are obtained. Eq. (2.15) admits the LB operator associated with the generator (2.16) iff

$$[X((u_3 + A(u,u_1)), X(u_5 + B(u,...,u_4))] = 0, \quad \text{(B1)}$$

or

$$(D^3 + A_1 + A_2D)(u_5 + B)$$
$$= (D^5 + B_1 + \sum_{j=2}^{4} B_jD^{j-1})(u_3 + A) = 0. \quad \text{(B2)}$$

Writing out $D^jA, D^jB, (j = 1, 2, 3, 4)$ and equating the coefficients of u_6 and u_5 in (B2) to zero, we obtain

$$\partial B/\partial u_4 = 0, \quad 3DB_4 = 5DA_2, \quad \text{(B3)}$$

or

$$B = \tfrac{5}{3}A_2u_3 + F(u,u_1,u_2). \quad \text{(B4)}$$

Replacing B in (B2) by (B4) and equating the coefficients of u_4 to zero, we obtain

$$F = \tfrac{5}{6}A_{22}u_2^2 + \tfrac{5}{3}A_{12}u_1u_2 + \tfrac{5}{3}A_1u_2 + g(u,u_1). \quad \text{(B5)}$$

Replacing B in (B2) by (B4), where F is given by (B5) and equating the coefficients of $u_3u_2^j$, $(j = 3, 2, 1, 0)$ to zero, we obtain, respectively,

$$A_{222} = 0, \quad A_{2221} = 0,$$
$$3g_{22} = 5A_2A_{22} + 10A_{112}u_1 + 5A_{11} + 5A_{1122}u_1^2, \quad \text{(B6)}$$
$$3g_{12}u_1 = \tfrac{5}{3}AA_{12} + \tfrac{5}{3}A_1A_{22}u_1 + \tfrac{10}{3}A_2A_{12}u_1$$
$$+ \tfrac{10}{3}A_{1112}u_1^3 + 5A_{111}u_1^2.$$

Equating the coefficients of u_2^3, u_2^2 in (B2) to zero, we obtain

$$A_{2211} = 0, \quad (A_2A_{221} - A_1A_{222})u_1 + A_1A_{22} - AA_{221} = 0. \quad \text{(B7)}$$

Solving Eqs. (B6a), (B6b), and (B7a), we obtain

$$A = \alpha u_1^3 + \gamma uu_1^2 + \beta u_1^2 + b(u)u_1 + c(u). \quad \text{(B8)}$$

Eq. (B7b) gives $\gamma = 0, \beta b' = 0, \beta c' = 0$. Therefore,

$$A = \alpha u_1^3 + \beta u_1^2 + b(u)u_1 + c(u), \quad \text{(B9)}$$
$$B = \tfrac{5}{3}A_2u_3 + \tfrac{5}{6}A_{22}u_2^2 + \tfrac{5}{3}A_{12}u_1u_2$$
$$+ \tfrac{5}{3}A_1u_2 + g(u,u_1), \quad \text{(B10)}$$

where

$$\beta b' = 0, \quad \beta c' = 0, \quad \text{(B11)}$$

and we have still to satisfy the compatibility equation of (B6c) and (B6d)

(i) $\beta \neq 0$. Then Eqs. (B9) and (B10) indicate that

$$A = \alpha u_1^3 + \beta u_1^2 + \gamma u_1 + \delta. \quad \text{(B12)}$$

The compatibility equation of (B6c) and (B6d) is then satisfied and, by integrating them, we obtain

$$g = \tfrac{5}{6}\int (A_2)^2 du_1. \quad \text{(B13)}$$

Replacing g in (B10) by (B13), where A is given by (B12), we obtain (2.17b).

(ii) $\beta = 0$. Then Eqs. (B9) and (B10) indicate that

$$A = \alpha u_1^3 + bu_1 + c. \quad \text{(B14)}$$

The compatibility equation of (B6c) and (B6d) gives

$$b''' + 8\alpha b' = 0, \quad \alpha c' = cb' = 0. \quad \text{(B15)}$$

If $c \neq 0$ we obtain trivial results, therefore

$$A = \alpha u_1^3 + b(u)u_1, \quad \text{(B16)}$$

where b satisfies (2.18b). Integrating (B6c), (B6d) we obtain g and then, using (B10), we obtain the generator (2.18c).

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